SNSB
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Ergodic Theory and Additive Combinatorics
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Seminar 3

(S3.1)

(i) $(X, 1_X)$ is minimal if and only if $|X| = 1$.

(ii) If (X, T) is minimal, then T is surjective.

(iii) A factor of a minimal TDS is also minimal.

(iv) If a product TDS is minimal, then so are each of its components.

(v) If (X_1, T_{X_1}), (X_2, T_{X_2}) are two minimal subsystems of a TDS (X, T), then either $X_1 \cap X_2 = \emptyset$ or $X_1 = X_2$.

(vi) A disjoint union of two TDSs is never a minimal TDS.

Proof. (i) Remark that for all $x \in X$, $\overline{\mathcal{O}}_+(x) = \{x\}$.

(ii) By Corollary 1.3.10, there exists a nonempty closed set $B \subseteq X$ such that $T(B) = B$. Since (X, T) is minimal, we must have $B = X$.

(iii) Let (X, T) be minimal and $\varphi : (X, T) \to (Y, S)$ be a surjective homomorphism. Assume $\emptyset \neq A \subseteq Y$ is a nonempty closed S-invariant subset of Y. We have to prove that $A = Y$. Let $B := \varphi^{-1}(A) \subseteq X$. Then B is closed and nonempty, since φ is continuous and surjective. Furthermore,

$$\begin{aligned} T(B) &= T(\varphi^{-1}(A)) = \{Tx \mid \varphi(x) \in A\} \subseteq \{Tx \mid (S \circ \varphi)(x) \in A\} \\ &\quad \text{since } (S \circ \varphi)(x) \in S(A) \subseteq A \\ &= \{Tx \mid (\varphi \circ T)(x) \in A\} \subseteq \varphi^{-1}(A) = B. \end{aligned}$$

Thus, B is a nonempty closed T-invariant subset of X, so we must have $B = X$. Using again the surjectivity of φ, it follows that

$$Y = \varphi(X) = \varphi(\varphi^{-1}(A)) = A.$$

(iv) By (iii) and Proposition 1.3.12.(ii).

(v) We have that X_1, X_2 are nonempty closed T-invariant subsets of X. Let $Y := X_1 \cap X_2$. Then Y is a closed T_{X_1}-invariant subset of X_1 (resp. a closed T_{X_2}-invariant subset of X_2), hence from minimality we must have $Y = \emptyset$ or $Y = X_1 = X_2$.

(vi) By Lemma 1.3.15.(i).

□

(S3.2) Let (X, T) be a TDS and assume that X is metrizable. For any $x \in X$, the following are equivalent:

(i) x is recurrent.

(ii) $\lim_{k\to\infty} T^{n_k}x = x$ for some sequence (n_k) in $\mathbb{Z}_+$.

(iii) $\lim_{k\to\infty} T^{n_k}x = x$ for some sequence (n_k) in $\mathbb{Z}_+$ such that $\lim_{k\to\infty} n_k = \infty$.

Proof. $(iii) \Rightarrow (ii)$ Obviously.
$(ii) \Rightarrow (i)$ Let U be an open neighborhood of x. Since $\lim_{k\to\infty} T^{n_k}x = x$, there exists $K \in \mathbb{Z}_+$ such that $T^{n_k}x \in U$ for all $k \geq K$.
$(i) \Rightarrow (iii)$ Use the fact that x is infinitely recurrent, by Proposition 1.6.3. Then $S_k := rt\left(x, B_{1/k}(x)\right)$ is an infinite set for every $k \geq 1$. Define $n_1 := \min S_1$, $n_{k+1} := \min S_{k+1} \setminus \{n_k\}$. Then (n_k) is a strictly increasing sequence of positive integers, so $\lim_{k\to\infty} n_k = \infty$. Furthermore, $d(x, T^{n_k}x) < 1/k$ for all $k \geq 1$, hence $\lim_{k\to\infty} T^{n_k}x = x$. □

(S3.3)

(i) If $\varphi : (X, T) \to (Y, S)$ is a homomorphism of TDSs and $x \in X$ is recurrent (almost periodic) in (X, T), then $\varphi(x)$ is recurrent (almost periodic) in (Y, S).

(ii) If (A, T_A) is a subsystem of (X, T) and $x \in A$, then x is recurrent (almost periodic) in (X, T) if and only if x is recurrent (almost periodic) in (A, T_A).

Proof. (i) Let V be an open neighborhood of $\varphi(x)$. Since φ is continuous, there exists an open neighborhood U of x such that $\varphi(U) \subseteq V$.

(a) As x is recurrent in (X, T), we have that $T^n x \in U$ for some $n \geq 1$. We get that

$$S^n(\varphi(x)) = \varphi(T^n x) \in \varphi(U) \subseteq V.$$

It follows that $\varphi(x)$ is recurrent in (X, T).

(b) As x is almost periodic in (X,T), we have that there exists $N \geq 1$ such that for all $m \geq 1$ there exists $k \in [m, m+N]$ such that $T^k x \in U$. We get that

$$S^k(\varphi(x)) = \varphi(T^k x) \in \varphi(U) \subseteq V.$$

It follows that $\varphi(x)$ is almost periodic in (X,T).

(ii) $\Leftarrow$ Use (i) and the fact the inclusion $j_A : (A, T_A) \to (X,T)$ is a homomorphism.
$\Rightarrow$ If U is an open neighborhood of x in A, then $U = A \cap V$, where V is an open neighborhood of x in X.

(a) If x is recurrent in (X,T), we have that $T^n x \in V$ for some $n \geq 1$. It follows that $T_A^n x = T^n x \in A \cap V = U$. Thus, x is recurrent in (A, T_A).

(b) If x is almost periodic in (X,T), we have that there exists $N \geq 1$ such that for all $m \geq 1$ there exists $k \in [m, m+N]$ such that $T^k x \in V$. Conclude as above that $T_A^k x = T^k x \in U$. Thus, x is almost periodic in (A, T_A).

□

(S3.4) Let (X,T) be a TDS. The following are equivalent:

(i) (X,T) is minimal.

(ii) every point of X is forward transitive and almost periodic.

(iii) there exists a forward transitive point $x_0 \in X$ which is also almost periodic.

Proof. $(i) \Rightarrow (ii)$ Apply Propositions 1.5.3.(ii) and 1.6.9.
$(ii) \Rightarrow (iii)$ Obviously.
$(iii) \Rightarrow (i)$ Let x_0 be a forward transitive point, hence $\overline{\mathcal{O}}_+(x) = X$. Since x_0 is almost periodic, we can apply (S4.2) to conclude that (X,T) is minimal. □

(S3.5) Let (X,T) be a TDS and $x \in X$. The following are equivalent:

(i) x is almost periodic.

(ii) For any open neighborhood U of x, there exists $N \geq 1$ such that

$$\mathcal{O}_+(x) \subseteq \bigcup_{k=0}^{N} T^{-k}(U).$$

(iii) $(\overline{\mathcal{O}}_+(x), T_{\overline{\mathcal{O}}_+(x)})$ is a minimal subsystem.

Proof. $(i) \Rightarrow (ii)$ We have obviously that $x \in U = T^0(U)$, so let $T^m x$ with $m \geq 1$. Since $rt(x, U)$ is syndetic, it follows that there exists $N \geq 1$ such that $rt(x, U) \cap [m, m + M] \neq \emptyset$ for all $m \geq 1$. Thus, there exists $p \in [m, m + N]$ such that $T^p x \in U$. Letting $k := p - m \in [0, N]$, we get that $T^k(T^m x) = T^p x \in U$, hence $T^m x \in T^{-k}(U)$.
$(ii) \Rightarrow (iii)$ We shall prove that $\mathcal{O}_+(y)$ is dense in $\overline{\mathcal{O}}_+(x)$ for every $y \in \overline{\mathcal{O}}_+(x)$, and then apply Proposition 1.5.3 to conclude minimality. It suffices to show that $x \in \overline{\mathcal{O}}_+(y)$. Let U be an open neighborhood of x. Then, by B.10.7.(i), there exists an open neighborhood V of x such that $\overline{V} \subseteq U$. By (ii), we have an $N \geq 1$ such that

$$\mathcal{O}_+(x) \subseteq \bigcup_{k=0}^{N} T^{-k}(V) \subseteq \bigcup_{k=0}^{N} T^{-k}(\overline{V}).$$

It follows that

$$y \in \overline{\mathcal{O}}_+(x) \subseteq \bigcup_{k=0}^{N} T^{-k}(\overline{V}) \subseteq \bigcup_{k=0}^{N} T^{-k}(U).$$

This implies $T^k y \in U$ for some $k = 0, \ldots, N$. Thus, $\mathcal{O}_+(y) \cap U \neq \emptyset$ for any open neighborhood U of x, that is $x \in \overline{\mathcal{O}}_+(y)$.
$(iii) \Rightarrow (i)$ Apply Proposition 1.6.9 and Lemma 1.6.7.(ii). □